THE TERWILLIGER ALGEBRA OF A DISTANCE-REGULAR GRAPH OF NEGATIVE TYPE

Štefko Miklavič*
Primorska Institute for Natural Science and Technology
University of Primorska
6000 Koper, Slovenia
stefko.miklavic@upr.si

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Abstract

Let Γ denote a distance-regular graph with diameter $D \geq 3$. Assume Γ has classical parameters (D, b, α, β) with $b < -1$. Let X denote the vertex set of Γ and let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. Fix $x \in X$ and let $A^* \in \mathrm{Mat}_X(\mathbb{C})$ denote the corresponding dual adjacency matrix. Let T denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, A^*. We call T the *Terwilliger algebra* of Γ with respect to x. We show that up to isomorphism there exist exactly two irreducible T-modules with endpoint 1; their dimensions are D and $2D-2$. For these T-modules we display a basis consisting of eigenvectors for A^*, and for each basis we give the action of A.

1 Introduction

Let Γ denote a Q-polynomial distance-regular graph with diameter $D \geq 3$ and intersection numbers a_i, b_i, c_i (see Section 2 for formal definitions). We recall the Terwilliger algebra of Γ. Let X denote the vertex set of Γ and let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. Fix a "base vertex" $x \in X$ and let $A^* \in \mathrm{Mat}_X(\mathbb{C})$ denote the corresponding dual adjacency matrix. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, A^*. The algebra T is called the *Terwilliger algebra* of Γ with respect to x [28]. T is closed under the conjugate-transpose map so T is semi-simple [28, Lemma 3.4(i)]. Therefore each T-module is a direct sum of irreducible T-modules. Describing the irreducible T-modules is an active area of research [3–17], [21, 26, 28, 31].

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In this description there is an important parameter called the *endpoint* which we now recall. Let W denote an irreducible T-module. By the *endpoint* of W we mean $\min\{i \mid 0 \le i \le D,\ E_i^* W \neq 0\}$, where $E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ is the projection onto the ith subconstituent of Γ with respect to x [28, p. 378]. There exists a unique irreducible T-module with endpoint 0 [12, Proposition 8.4]; for a detailed description see [7, 12].

Consider now the irreducible T-modules with endpoint 1. If Γ is bipartite, then these T-modules are described in [7, 8]. If Γ is nonbipartite with $a_1 = 0$, then these T-modules are described in [4, 21]. For the rest of this Introduction assume $a_1 \neq 0$. Assume further that Γ is of negative type and not a near polygon. In [22] we described the combinatorial structure of Γ. In the present paper we use this description to obtain the irreducible T-modules that have endpoint 1. To summarize our results we note the following. Let W denote an irreducible T-module with endpoint 1. Observe that $E_1^* W$ is a 1-dimensional eigenspace for $E_1^* A E_1^*$ [16, Theorem 2.2]. The corresponding eigenvalue is called the *local eigenvalue* of W. We show that up to isomorphism there exist exactly two irreducible T-modules with endpoint 1. The first one has dimension D and local eigenvalue -1. The second one has dimension $2D-2$ and local eigenvalue a_1. For these modules we display a basis consisting of eigenvectors for A^*, and for each basis we give the action of A. At present there is no classification of graphs that satisfy our assumptions; see [22, Section 6] for a summary of what is known.

2 Preliminaries

In this section we review some definitions and basic results concerning distance-regular graphs. See the book of Brouwer, Cohen and Neumaier [2] for more background information.

Let $\mathbb{C}$ denote the complex number field and let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*. We endow V with the Hermitean inner product $\langle\, , \rangle$ that satisfies $\langle u, v\rangle = u^t \overline{v}$ for $u, v \in V$, where t denotes transpose and $\overline{}$ denotes complex conjugation. For $y \in X$ let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V. The following will be useful: for each $B \in \mathrm{Mat}_X(\mathbb{C})$ we have

$$\langle u, Bv\rangle = \langle \overline{B}^t u, v\rangle \qquad (u, v \in V). \tag{1}$$

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call D the *diameter* of Γ. For a vertex $x \in X$ and an integer i let $\Gamma_i(x)$ denote the set of vertices at distance i from x. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$ we say Γ is *regular with valency k*

whenever $|\Gamma(x)| = k$ for all $x \in X$. We say Γ is *distance-regular* whenever for all integers h, i, j $(0 \le h, i, j \le D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y. The p^h_{ij} are called the *intersection numbers* of Γ.

For the rest of this paper we assume Γ is distance-regular with diameter $D \ge 3$. Note that $p^h_{ij} = p^h_{ji}$ for $0 \le h, i, j \le D$. For convenience set $c_i := p^i_{1,i-1}\,(1 \le i \le D)$, $a_i := p^i_{1i}\,(0 \le i \le D)$, $b_i := p^i_{1,i+1}\,(0 \le i \le D-1)$, $k_i := p^0_{ii}\,(0 \le i \le D)$, and $c_0 = b_D = 0$. By the triangle inequality the following hold for $0 \le h, i, j \le D$: (i) $p^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two; (ii) $p^h_{ij} \ne 0$ if one of h, i, j equals the sum of the other two. In particular $c_i \ne 0$ for $1 \le i \le D$ and $b_i \ne 0$ for $0 \le i \le D-1$. We observe that Γ is regular with valency $k = k_1 = b_0$ and that

$$c_i + a_i + b_i = k \qquad (0 \le i \le D). \tag{2}$$

Note that $k_i = |\Gamma_i(x)|$ for $x \in X$ and $0 \le i \le D$. By [2, p. 127],

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad (0 \le i \le D). \tag{3}$$

We recall the Bose-Mesner algebra of Γ. For $0 \le i \le D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \ne i \end{cases} \qquad (x, y \in X). \tag{4}$$

We call A_i the ith *distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. We observe (ai) $A_0 = I$; (aii) $\sum_{i=0}^{D} A_i = J$; (aiii) $\overline{A_i} = A_i$ $(0 \le i \le D)$; (aiv) $A_i^t = A_i$ $(0 \le i \le D)$; (av) $A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h$ $(0 \le i, j \le D)$, where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_X(\mathbb{C})$. Using these facts we find $A_0, A_1, \ldots, A_D$ is a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. We call M the *Bose-Mesner algebra* of Γ. It turns out that A generates M [1, p. 190]. By [2, p. 45], M has a second basis $E_0, E_1, \ldots, E_D$ such that (ei) $E_0 = |X|^{-1} J$; (eii) $\sum_{i=0}^{D} E_i = I$; (eiii) $\overline{E_i} = E_i$ $(0 \le i \le D)$; (eiv) $E_i^t = E_i$ $(0 \le i \le D)$; (ev) $E_i E_j = \delta_{ij} E_i$ $(0 \le i, j \le D)$. We call $E_0, E_1, \ldots, E_D$ the *primitive idempotents* of Γ.

We now recall the Krein parameters. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{C})$. Observe $A_i \circ A_j = \delta_{ij} A_i$ for $0 \le i, j \le D$, so M is closed under $\circ$. Thus there exist complex scalars q^h_{ij} $(0 \le h, i, j \le D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \qquad (0 \le i, j \le D).$$

By [2, Proposition 4.1.5], q^h_{ij} is real and nonnegative for $0 \le h, i, j \le D$. The q^h_{ij} are called the *Krein parameters of* Γ. The graph Γ is said to be *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever for $0 \le h, i, j \le D$,

$q^h_{ij} = 0$ (resp. $q^h_{ij} \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two. For the rest of this section assume Γ is Q-polynomial with respect to $E_0, E_1, \ldots, E_D$.

We now recall the dual idempotents of Γ. To do this fix a vertex $x \in X$. We view x as a "base vertex". For $0 \leq i \leq D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E^*_i)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \qquad (y \in X). \tag{5}$$

We call E^*_i the ith *dual idempotent* of Γ with respect to x [28, p. 378]. We observe (i) $\sum_{i=0}^{D} E^*_i = I$; (ii) $\overline{E^*_i} = E^*_i$ $(0 \leq i \leq D)$; (iii) $E^{*t}_i = E^*_i$ $(0 \leq i \leq D)$; (iv) $E^*_i E^*_j = \delta_{ij} E^*_i$ $(0 \leq i, j \leq D)$. By these facts $E^*_0, E^*_1, \ldots, E^*_D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose-Mesner algebra* of Γ with respect to x [28, p. 378]. For $0 \leq i \leq D$ we have

$$E^*_i V = \mathrm{Span}\{\hat{y} \mid y \in X,\ \partial(x, y) = i\}$$

so $\dim E^*_i V = k_i$. We call $E^*_i V$ the ith *subconstituent of* Γ with respect to x. Note that

$$V = E^*_0 V + E^*_1 V + \cdots + E^*_D V \qquad \text{(orthogonal direct sum)}. \tag{6}$$

Moreover E^*_i is the projection from V onto $E^*_i V$ for $0 \leq i \leq D$. Let $A^* = A^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$A^*_{yy} = |X| E_{xy} \qquad (y \in X),$$

where $E = E_1$. We call A^* the *dual adjacency matrix of* Γ with respect to x. By [28, Lemma 3.11(ii)] A^* generates M^*.

We recall the Terwilliger algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. We call T the *Terwilliger algebra of* Γ with respect to x [28, Definition 3.3]. Recall M (resp. M^*) is generated by A (resp. A^*) so T is generated by A, A^*. We observe T has finite dimension. By construction T is closed under the conjugate-transpose map so T is semi-simple [28, Lemma 3.4(i)].

By a T-*module* we mean a subspace W of V such that $BW \subseteq W$ for all $B \in T$. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than 0 and W. Assume W is irreducible. Then A and A^* act on W as a tridiagonal pair [17, Example 1.4]. We refer the reader to [17, 18, 19, 20, 24, 25] and the references therein for background on tridiagonal pairs.

By [14, Corollary 6.2] any T-module is an orthogonal direct sum of irreducible T-modules. In particular the standard module V is an orthogonal direct sum of irreducible T-modules. Let W, W' denote T-modules. By an *isomorphism of* T-*modules* from W to W' we mean an isomorphism of vector spaces $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. The T-modules W, W' are said to be *isomorphic* whenever there exists an isomorphism of T-modules from W to W'. By [7, Lemma 3.3] any two nonisomorphic irreducible T-modules are orthogonal. Let W denote an irreducible T-module. By [28, Lemma 3.4(iii)] W is an orthogonal direct sum of the nonvanishing spaces among

$E_0^*W, E_1^*W, \ldots, E_D^*W$. By the *endpoint* of W we mean $\min\{i \mid 0 \le i \le D,\ E_i^*W \ne 0\}$. By the *diameter* of W we mean $|\{i \mid 0 \le i \le D,\ E_i^*W \ne 0\}| - 1$.

By [12, Proposition 8.3, Proposition 8.4] $M\hat{x}$ is the unique irreducible T-module with endpoint 0 and the unique irreducible T-module with diameter D. Moreover $M\hat{x}$ is the unique irreducible T-module on which E_0 does not vanish. We call $M\hat{x}$ the *primary module*.

We finish this section with some comments on local eigenvalues. Let $\Delta = \Delta(x)$ denote the vertex-subgraph of Γ induced on the set of vertices in X adjacent x, and let $\breve{A}$ denote the adjacency matrix of Δ. By the *local eigenvalues of* Γ we mean the eigenvalues of $\breve{A}$. Note that the local eigenvalues of Γ are precisely the eigenvalues of $E_1^*AE_1^*$ on E_1^*V.

Let W denote an irreducible T-module with endpoint 1. By [16, Theorem 2.2] E_1^*W is a one-dimensional eigenspace for $E_1^*AE_1^*$; we call the corresponding eigenvalue the *local eigenvalue* of W.

3 Distance-regular graphs of negative type

In this section we recall what it means for Γ to have classical parameters and negative type. The graph Γ is said to have *classical parameters* (D, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \Big(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\Big) \qquad (1 \le i \le D),$$

$$b_i = \Big(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\Big)\Big(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\Big) \qquad (0 \le i \le D-1),$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{j-1}.$$

In this case b is an integer and $b \notin \{0, -1\}$. If Γ has classical parameters then Γ is Q-polynomial [2, Corollary 8.4.2]. We say that Γ has *negative type* whenever Γ has classical parameters (D, b, α, β) such that $b < -1$.

We now recall kites and parallelograms. Fix an integer i $(2 \le i \le D)$. By a *kite of length* i (or i*-kite*) in Γ we mean a 4-tuple $uvwz$ of vertices of Γ such that u, v, w are mutually adjacent, and $\partial(u,z) = i$, $\partial(v,z) = \partial(w,z) = i-1$. By a *parallelogram of length* i (or i*-parallelogram*) in Γ we mean a 4-tuple $uvwz$ of vertices of Γ such that $\partial(u,v) = \partial(w,z) = 1$, $\partial(u,z) = i$, and $\partial(v,z) = \partial(u,w) = \partial(v,w) = i-1$. By [27, Theorem 2.12] and [22, Theorem 4.2], if Γ has negative type then Γ has no parallelograms or kites of any length.

We now recall the near polygons. The graph Γ is called a *near polygon* whenever $a_i = a_1c_i$ for $1 \le i \le D-1$ and Γ has no 2-kite [24]. From now on we adopt the following notational convention.

Notation 3.1 Let $\Gamma = (X, R)$ denote a distance-regular graph with classical parameters (D, b, α, β), $D \ge 3$, with valency k and $a_1 \ne 0$. Assume that Γ is of negative type and Γ is

not a near polygon. Let $A_0, A_1, \ldots, A_D$ denote the distance matrices of Γ, and let V denote the standard module of Γ. We fix $x \in X$ and let $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, and $T = T(x)$ denote the corresponding dual idempotents and Terwilliger algebra, respectively.

The following result is an immediate consequence of [22, Lemma 6.4, Lemma 6.5].

Corollary 3.2 *With reference to Notation 3.1 we have $a_i > a_1 c_i$ for $2 \le i \le D$.*

4 The sets D_j^i

With reference to Notation 3.1, in this section we define certain subsets D_j^i of X and explore their properties.

Definition 4.1 With reference to Notation 3.1 fix $z \in \Gamma(x)$. For all integers i, j we define $D_j^i = D_j^i(x, z)$ by

$$D_j^i = \Gamma_i(x) \cap \Gamma_j(z).$$

We observe $D_j^i = \emptyset$ unless $0 \le i, j \le D$.

Lemma 4.2 *With reference to Notation 3.1 and Definition 4.1 the following* (i), (ii) *hold for $0 \le i, j \le D$.*

(i) $|D_j^i| = p_{ij}^1$.

(ii) $D_j^i = \emptyset$ *if and only if* $p_{ij}^1 = 0$.

PROOF. (i) Immediate from the definition of p_{ij}^1 and D_j^i.
(ii) Immediate from (i) above. ∎

Lemma 4.3 ([2, p. 134]) *With reference to Notation 3.1 the following* (i), (ii) *hold.*

(i) $p_{i-1,i}^1 = p_{i,i-1}^1 = c_i k_i k^{-1}$ $(1 \le i \le D)$.

(ii) $p_{ii}^1 = a_i k_i k^{-1}$ $(0 \le i \le D)$.

Lemma 4.4 *With reference to Notation 3.1 the following* (i)–(iii) *hold.*

(i) $p_{i-1,i}^1 \ne 0$, $p_{i,i-1}^1 \ne 0$ $(1 \le i \le D)$.

(ii) $p_{00}^1 = 0$, $p_{ii}^1 \ne 0$ $(1 \le i \le D)$.

(iii) $p_{ij}^1 = 0$ *if* $|i - j| \notin \{0, 1\}$ $(0 \le i, j \le D)$.

PROOF. (i) Immediate from Lemma 4.3(i).
(ii) It is clear that $p_{00}^1 = 0$ and $p_{11}^1 = a_1 \ne 0$. Assume $2 \le i \le D$. By Corollary 3.2 we find $a_i \ne 0$ so $p_{ii}^1 \ne 0$ in view of Lemma 4.3(ii).
(iii) Immediate from the triangle inequality. ∎

Lemma 4.5 *With reference to Notation 3.1 and Definition 4.1 the following* (i)–(iii) *hold.*

(i) $\partial(u, y) = 1$ *for all distinct* $u, y \in D_1^1$.

(ii) *There are no edges between* $D_i^{i-1} \cup D_{i-1}^i$ *and* D_{i-1}^{i-1} *for* $2 \leq i \leq D$.

(iii) *For* $1 \leq i \leq D$ *we have* $\partial(u, y) = i$ *for all* $u \in D_1^1$ *and all* $y \in D_i^{i-1} \cup D_{i-1}^i$.

Proof. (i) If u, y are not adjacent, then $yxzu$ is a 2-kite, a contradiction.
(ii) There does not exist adjacent vertices v, w with $v \in D_{i-1}^i$ and $w \in D_{i-1}^{i-1}$; otherwise $xzwv$ is an i-parallelogram, a contradiction. A similar argument shows that there does not exist adjacent vertices v, w with $v \in D_i^{i-1}$ and $w \in D_{i-1}^{i-1}$.
(iii) If $i = 1$ then the result is clear. Assume $2 \leq i \leq D$. By the triangle inequality we find $\partial(u, y) \in \{i - 1, i\}$. But $\partial(u, y) \geq i$ by (ii) above, and the result follows. ∎

We end this section with two remarks on the local eigenvalues.

Corollary 4.6 *With reference to Notation 3.1 let* $\Delta = \Delta(x)$ *denote the vertex-subgraph of* Γ *induced on the set of vertices in* X *adjacent* x. *Then the following* (i), (ii) *hold.*

(i) Δ *is a disjoint union of* $k(a_1 + 1)^{-1}$ *cliques, each consisting of* $a_1 + 1$ *vertices.*

(ii) *The local eigenvalues of* Γ *are* a_1 *with multiplicity* $k(a_1 + 1)^{-1}$, *and* -1 *with multiplicity* $ka_1(a_1 + 1)^{-1}$.

Proof. (i) Immediate from Lemma 4.5(i),(ii).
(ii) Immediate from (i) above. ∎

Corollary 4.7 *With reference to Notation 3.1 let* W *denote an irreducible* T*-module with endpoint* 1. *Then the local eigenvalue of* W *is* a_1 *or* -1.

Proof. The local eigenvalue of W is a local eigenvalue of Γ. The result now follows from Corollary 4.6(ii). ∎

5 The sets $D_i^i(0)$ and $D_i^i(1)$

With reference to Notation 3.1, in this section we define certain subsets $D_i^i(0)$ and $D_i^i(1)$ of X and explore their properties.

Lemma 5.1 *With reference to Notation 3.1 and Definition 4.1, for* $1 \leq i \leq D$ *and* $y \in D_i^i$ *we have* $|\Gamma_{i-1}(y) \cap D_1^1| \leq 1$.

Proof. Assume that $|\Gamma_{i-1}(y) \cap D_1^1| \geq 2$ and pick distinct $u, v \in \Gamma_{i-1}(y) \cap D_1^1$. Then $xuvy$ is an i-kite, a contradiction. ∎

Definition 5.2 With reference to Notation 3.1 and Definition 4.1, for an integer i and $j \in \{0, 1\}$ define a set $D_i^i(j) = D_i^i(j)(x, z)$ by

$$D_i^i(j) = \{y \in D_i^i \mid |\Gamma_{i-1}(y) \cap D_1^1| = j\}.$$

We observe $D_i^i(j) = \emptyset$ unless $1 \le i \le D$. By Lemma 5.1 D_i^i is the disjoint union of $D_i^i(1)$ and $D_i^i(0)$.

Lemma 5.3 ([22, Lemma 6.4]) *With reference to Notation 3.1 and Definition 5.2 the following* (i), (ii) *hold for* $1 \le i \le D$.

(i) $|D_i^i(1)| = a_1 c_i k_i k^{-1}$.

(ii) $|D_i^i(0)| = (a_i - a_1 c_i) k_i k^{-1}$.

Lemma 5.4 *With reference to Notation 3.1 and Definition 5.2 the following* (i), (ii) *hold for* $1 \le i \le D$.

(i) $D_i^i(1) \neq \emptyset$ *for* $1 \le i \le D$.

(ii) $D_i^i(0) \neq \emptyset$ *for* $2 \le i \le D$ *and* $D_1^1(0) = \emptyset$.

Proof. Combine Corollary 3.2 and Lemma 5.3. ∎

Lemma 5.5 ([22, Lemma 4.4(i)]) *With reference to Notation 3.1, Definition 4.1 and Definition 5.2, for* $2 \le i \le D$ *we have* $\partial(u, y) = i$ *for all* $u \in D_1^1$ *and all* $y \in D_i^i(0)$.

Lemma 5.6 ([22, Sections 5,6]) *With reference to Notation 3.1, Definition 4.1 and Definition 5.2 the following* (i)–(iii) *hold.*

(i) *For* $1 \le i \le D$, *each vertex in* D_{i-1}^i *(resp.* D_i^{i-1}*) is adjacent to*

precisely	c_{i-1}	*vertices in* D_{i-2}^{i-1} *(resp.* D_{i-1}^{i-2}*),*
precisely	$c_i - c_{i-1}$	*vertices in* D_i^{i-1} *(resp.* D_{i-1}^i*),*
precisely	a_{i-1}	*vertices in* D_{i-1}^i *(resp.* D_i^{i-1}*),*
precisely	b_i	*vertices in* D_i^{i+1} *(resp.* D_{i+1}^i*),*
precisely	$a_1(c_i - c_{i-1})$	*vertices in* $D_i^i(1)$,
precisely	$a_i - a_{i-1} - a_1(c_i - c_{i-1})$	*vertices in* $D_i^i(0)$,

and no other vertices in X.

(ii) *For* $2 \le i \le D$, *each vertex in* $D_i^i(0)$ *is adjacent to*

precisely	$c_i(b^{i-2} - 1)(b^i - 1)^{-1}$	*vertices in* $D_{i-1}^{i-1}(0)$,
precisely	$a_1 c_i(b^i - b^{i-2})(b^i - 1)^{-1}$	*vertices in* $D_i^i(1)$,
precisely	$c_i(b^i - b^{i-2})(b^i - 1)^{-1}$	*vertices in* D_{i-1}^i,
precisely	$c_i(b^i - b^{i-2})(b^i - 1)^{-1}$	*vertices in* D_i^{i-1},
precisely	b_i	*vertices in* $D_{i+1}^{i+1}(0)$,
precisely	$a_i - c_i(a_1 + 1)(b^i - b^{i-2})(b^i - 1)^{-1}$	*vertices in* $D_i^i(0)$,

and no other vertices in X.

(iii) *For* $1 \le i \le D$*, each vertex in* $D^i_i(1)$ *is adjacent to*

precisely	c_{i-1}	*vertices in* $D^{i-1}_{i-1}(1)$,
precisely	$(a_1-1)(c_i-c_{i-1})+a_{i-1}$	*vertices in* $D^i_i(1)$,
precisely	c_i-c_{i-1}	*vertices in* D^i_{i-1},
precisely	c_i-c_{i-1}	*vertices in* D^{i-1}_i,
precisely	b_i	*vertices in* $D^{i+1}_{i+1}(1)$,
precisely	$a_i-a_{i-1}-a_1(c_i-c_{i-1})$	*vertices in* $D^i_i(0)$,

and no other vertices in X*.*

6 Some products in T

With reference to Notation 3.1, in this section we evaluate several products in T which we shall need later.

Lemma 6.1 *With reference to Notation 3.1, for* $0 \le h, i, j \le D$ *and* $y, z \in X$ *the* (y,z)*-entry of* $E^*_h A_i E^*_j$ *is* 1 *if* $\partial(x,y)=h$*,* $\partial(y,z)=i$*,* $\partial(x,z)=j$*, and* 0 *otherwise.*

Proof. Compute the (y,z)-entry of $E^*_h A_i E^*_j$ by matrix multiplication and simplify the result using (4) and (5). ∎

Corollary 6.2 ([28, Lemma 3.2]) *With reference to Notation 3.1,*

$$E^*_h A_i E^*_j = 0 \quad \text{if and only if} \quad p^h_{ij} = 0 \qquad\qquad (0 \le h, i, j \le D).$$

Proof. Immediate from Lemma 6.1. ∎

Corollary 6.3 *With reference to Notation 3.1 and Definition 4.1, for* $0 \le i, j \le D$ *and* $y \in X$ *the* (y,z)*-entry of* $E^*_i A_j E^*_1$ *is* 1 *if* $y \in D^i_j$*, and* 0 *otherwise.*

Proof. Immediate from Lemma 6.1 . ∎

Corollary 6.4 *With reference to Notation 3.1 the following* (i), (ii) *hold.*

(i) $E^*_i A_{i-1} E^*_1 + E^*_i A_i E^*_1 + E^*_i A_{i+1} E^*_1 = E^*_i J E^*_1$ *for* $1 \le i \le D-1$*.*

(ii) $E^*_D A_{D-1} E^*_1 + E^*_D A_D E^*_1 = E^*_D J E^*_1$*.*

Proof. For each equation evaluate the right-hand side using assertion (aii) below line (4), and simplify the result using Corollary 6.2 and assertion (i) above line (2). ∎

Lemma 6.5 *With reference to Notation 3.1, for* $0 \le h, i, j, r, s \le D$ *and* $y, z \in X$ *the* (y,z)*-entry of* $E^*_h A_r E^*_i A_s E^*_j$ *is* $|\Gamma_i(x) \cap \Gamma_r(y) \cap \Gamma_s(z)|$ *if* $\partial(x,y)=h$*,* $\partial(x,z)=j$*, and* 0 *otherwise.*

Proof. Compute the (y,z)-entry of $E^*_h A_r E^*_i A_s E^*_j$ by matrix multiplication and simplify the result using (4) and (5). ∎

Corollary 6.6 *With reference to Notation 3.1 and Definition 5.2, for $1 \leq i \leq D$ and $y \in X$ the following* (i), (ii) *hold.*

(i) *The (y, z)-entry of $E_i^* A_{i-1} E_1^* A E_1^*$ is 1 if $y \in D_i^i(1)$, and 0 otherwise.*

(ii) *The (y, z)-entry of $E_i^*(A_i - A_{i-1} E_1^* A) E_1^*$ is 1 if $y \in D_i^i(0)$, and 0 otherwise.*

Proof. (i) Immediate from Lemma 4.5(iii), Lemma 5.5 and Lemma 6.5.
(ii) By Corollary 6.3 the (y, z)-entry of $E_i^* A_i E_1^*$ is 1 if $y \in D_i^i$, and 0 otherwise. The result now follows from (i) above and since D_i^i is the disjoint union of $D_i^i(0)$ and $D_i^i(1)$. ∎

7 The matrices L, F, R

With reference to Notation 3.1, in this section we recall the matrices L, F, R and use them to interpret Theorem 5.6.

Definition 7.1 With reference to Notation 3.1 we define matrices $L = L(x)$, $F = F(x)$, $R = R(x)$ by

$$L = \sum_{h=1}^{D} E_{h-1}^* A E_h^*, \qquad F = \sum_{h=0}^{D} E_h^* A E_h^*, \qquad R = \sum_{h=0}^{D-1} E_{h+1}^* A E_h^*.$$

Note that $A = L + F + R$ [7, Lemma 4.4]. We call L, F, and R the *lowering matrix*, the *flat matrix*, and the *raising matrix of* Γ with respect to x.

Lemma 7.2 *With reference to Notation 3.1 and Definition 7.1 the following* (i)–(iii) *hold.*

(i) $LE_1^* = E_0^* A E_1^*$.

(ii) *For $2 \leq i \leq D$,*

$$LE_i^* A_{i-1} E_1^* = b_{i-1} E_{i-1}^* A_{i-2} E_1^* + (c_i - c_{i-1}) E_{i-1}^* A_i E_1^*.$$

(iii) *For $1 \leq i \leq D-1$,*

$$LE_i^* A_{i+1} E_1^* = b_i E_{i-1}^* A_i E_1^*.$$

Proof. For each equation and for $y, z \in X$ compute the (y, z)-entry of each side and interpret the results using Theorem 5.6, Corollary 6.3 and Lemma 6.5. ∎

Lemma 7.3 *With reference to Notation 3.1 and Definition 7.1 the following* (i)–(iii) *hold.*

(i) $FE_1^* = E_1^* A E_1^*$.

(ii) *For $2 \leq i \leq D$,*

$$\begin{aligned} FE_i^* A_{i-1} E_1^* &= a_{i-1} E_i^* A_{i-1} E_1^* + (c_i - c_{i-1}) E_i^* A_{i-1} E_1^* A E_1^* \\ &\quad + c_i (b^i - b^{i-2})(b^i - 1)^{-1} E_i^* (A_i - A_{i-1} E_1^* A) E_1^*. \end{aligned}$$

(iii) *For* $1 \le i \le D-1$,

$$FE_i^* A_{i+1} E_1^* = a_i E_i^* A_{i+1} E_1^*.$$

Proof. For each equation and for $y, z \in X$ compute the (y,z)-entry of each side and interpret the results using Theorem 5.6, Corollary 6.3, Lemma 6.5 and Corollary 6.6. ∎

Lemma 7.4 *With reference to Notation 3.1 and Definition 7.1 the following* (i)–(iv) *hold.*

(i) *For* $1 \le i \le D-1$,

$$RE_i^* A_{i-1} E_1^* = c_i E_{i+1}^* A_i E_1^*.$$

(ii) $RE_D^* A_{D-1} E_1^* = 0.$

(iii) *For* $1 \le i \le D-2$,

$$\begin{aligned} RE_i^* A_{i+1} E_1^* &= c_{i+1} E_{i+1}^* A_{i+2} E_1^* + (c_{i+1} - c_i) E_{i+1}^* A_i E_1^* \\ &\quad + (c_{i+1} - c_i) E_{i+1}^* A_i E_1^* A E_1^* \\ &\quad + c_{i+1} (b^{i+1} - b^{i-1})(b^{i+1} - 1)^{-1} E_{i+1}^* (A_{i+1} - A_i E_1^* A) E_1^*. \end{aligned}$$

(iv)

$$\begin{aligned} RE_{D-1}^* A_D E_1^* &= (c_D - c_{D-1}) E_D^* A_{D-1} E_1^* + (c_D - c_{D-1}) E_D^* A_{D-1} E_1^* A E_1^* \\ &\quad + c_D (b^D - b^{D-2})(b^D - 1)^{-1} E_D^* (A_D - A_{D-1} E_1^* A) E_1^*. \end{aligned}$$

Proof. For each equation and for $y, z \in X$ compute the (y,z)-entry of each side and interpret the results using Theorem 5.6, Corollary 6.3, Lemma 6.5 and Corollary 6.6. ∎

8 More products in T

With reference to Notation 3.1, in this section we evaluate more products in T which we will need later.

Lemma 8.1 *With reference to Notation 3.1, for* $y, z \in \Gamma(x)$ *and* $1 \le i \le D$ *the number* $|\Gamma_i(x) \cap \Gamma_{i-1}(y) \cap \Gamma_{i-1}(z)|$ *is equal to* $c_i k_i k^{-1}$ *if* $y = z$, 0 *if* $\partial(y,z) = 1$, *and* $c_i(c_i - 1) k_i k^{-1} b_1^{-1}$ *if* $\partial(y,z) = 2$.

Proof. If $y = z$ then the result follows by Lemma 4.3(i). If $\partial(y,z) = 1$ then the result follows by Lemma 4.5(iii). Assume $\partial(y,z) = 2$. Abbreviate $D_j^\ell = D_j^\ell(x,z)$ $(0 \le j, \ell \le D)$ and note that $y \in D_2^1$. It follows from Theorem 5.6 that the number of paths of length $i-1$ between y and D_{i-1}^i is independent of y. Moreover, between any two vertices of Γ which are at distance $i-1$, there exist exactly $c_1 c_2 \cdots c_{i-1}$ paths of length $i-1$. Therefore the scalar $|D_{i-1}^i \cap \Gamma_{i-1}(y)|$ is independent of y; denote this scalar by α_i. For $v \in D_{i-1}^i$ we have $|\Gamma_{i-1}(v) \cap \Gamma(x)| = c_i$, so using Lemma 4.5(iii) we find $|\Gamma_{i-1}(v) \cap D_2^1| = c_i - 1$. Using these comments we count in two ways the number of pairs (y, v) such that $y \in D_2^1$, $v \in D_{i-1}^i$, and $\partial(y, v) = i-1$. This yields $\alpha_i |D_2^1| = |D_{i-1}^i|(c_i - 1)$. Evaluating this equation using Lemma 4.2(i) and Lemma 4.3(i) we find $\alpha_i = c_i(c_i - 1) k_i k^{-1} b_1^{-1}$. The result follows. ∎

Corollary 8.2 *With reference to Notation 3.1, for $1 \le i \le D$ we have*

$$E_1^* A_{i-1} E_i^* A_{i-1} E_1^* = c_i k_i k^{-1} E_1^* + c_i(c_i - 1) k_i k^{-1} b_1^{-1} E_1^* A_2 E_1^*.$$

Proof. For $y, z \in X$ we show that the (y,z)-entry of both sides are equal. If $y \notin \Gamma(x)$ or $z \notin \Gamma(x)$ then the (y,z)-entry of each side is 0. If $y, z \in \Gamma(x)$ then the (y,z)-entry of both sides are equal by Corollary 6.3, Lemma 6.5 and Lemma 8.1. The result follows. ∎

Lemma 8.3 *With reference to Notation 3.1, for $y, z \in \Gamma(x)$ and $1 \le i \le D-1$ the number $|\Gamma_i(x) \cap \Gamma_{i-1}(y) \cap \Gamma_{i+1}(z)|$ is equal to 0 if $y = z$, 0 if $\partial(y,z) = 1$, and $c_i b_i k_i k^{-1} b_1^{-1}$ if $\partial(y,z) = 2$.*

Proof. If $y = z$ then the result is clear. If $\partial(y,z) = 1$ then the result follows by Lemma 4.5(iii). Assume $\partial(y,z) = 2$. Abbreviate $D_j^\ell = D_j^\ell(x,z)$ $(0 \le j, \ell \le D)$ and note that $y \in D_2^1$. It follows from Theorem 5.6 that the number of paths of length $i-1$ between y and D_{i+1}^i is independent of y. Moreover, between any two vertices of Γ which are at distance $i-1$, there exist exactly $c_1 c_2 \cdots c_{i-1}$ paths of length $i-1$. Therefore the scalar $|D_{i+1}^i \cap \Gamma_{i-1}(y)|$ is independent of y; denote this scalar by α_i. For $v \in D_{i+1}^i$ we have $|\Gamma_{i-1}(v) \cap \Gamma(x)| = c_i$, so using Lemma 4.5(iii) we find $|\Gamma_{i-1}(v) \cap D_2^1| = c_i$. Using these comments we count in two ways the number of pairs (y,v) such that $y \in D_2^1$, $v \in D_{i+1}^i$, and $\partial(y,v) = i-1$. This yields $\alpha_i |D_2^1| = |D_{i+1}^i| c_i$. Evaluating this equation using Lemma 4.2(i), Lemma 4.3(i) and $c_{i+1}k_{i+1} = b_i k_i$ we find $\alpha_i = c_i b_i k_i k^{-1} b_1^{-1}$. The result follows. ∎

Corollary 8.4 *With reference to Notation 3.1, for $1 \le i \le D-1$ we have*

$$E_1^* A_{i-1} E_i^* A_{i+1} E_1^* = c_i b_i k_i k^{-1} b_1^{-1} E_1^* A_2 E_1^*.$$

Proof. For $y, z \in X$ we show that the (y,z)-entry of both sides are equal. If $y \notin \Gamma(x)$ or $z \notin \Gamma(x)$ then the (y,z)-entry of each side is 0. If $y, z \in \Gamma(x)$ then the (y,z)-entry of both sides are equal by Corollary 6.3, Lemma 6.5 and Lemma 8.3. The result follows. ∎

Lemma 8.5 *With reference to Notation 3.1, for $y, z \in \Gamma(x)$ and $1 \le i \le D-1$ the number $|\Gamma_i(x) \cap \Gamma_{i+1}(y) \cap \Gamma_{i+1}(z)|$ is equal to $b_i k_i k^{-1}$ if $y = z$, $b_i k_i k^{-1}$ if $\partial(y,z) = 1$, and $b_i(b_1 - a_i - c_i) k_i k^{-1} b_1^{-1}$ if $\partial(y,z) = 2$.*

Proof. If $y = z$ the result follows by Lemma 4.3(i) and since $c_{i+1}k_{i+1} = b_i k_i$. If $\partial(y,z) = 1$ then the result follows by Lemma 4.3(i), Lemma 4.5(iii) and since $c_{i+1}k_{i+1} = b_i k_i$. Assume $\partial(y,z) = 2$. Abbreviate $D_j^\ell = D_j^\ell(x,z)$ $(0 \le j, \ell \le D)$ and note that $y \in D_2^1$. We first claim that $|D_{i+1}^i \cap \Gamma_i(y)| = a_i b_i k_i k^{-1} b_1^{-1}$. It follows from Theorem 5.6 that the number of paths of length i between y and D_{i+1}^i is independent of y. Moreover, between any two vertices of Γ which are at distance $i-1$ (i, respectively), there exist exactly $(a_1 + a_2 + \cdots + a_{i-1}) c_1 c_2 \cdots c_{i-1}$ ($c_1 c_2 \cdots c_i$, respectively) paths of length i. By this and Lemma 8.3 the scalar $|D_{i+1}^i \cap \Gamma_i(y)|$ is independent of y; denote this scalar by α_i. For $v \in D_{i+1}^i$ we have $|\Gamma_i(v) \cap \Gamma(x)| = a_i$, so using Lemma 4.5(iii) we find $|\Gamma_i(v) \cap D_2^1| = a_i$. Using these comments we count in two ways the number of pairs (y,v) such that $y \in D_2^1$,

$v \in D^i_{i+1}$, and $\partial(y,v) = i$. This yields $\alpha_i |D^1_2| = |D^i_{i+1}|\, a_i$. Evaluating this equation using Lemma 4.2(i), Lemma 4.3(i) and $c_{i+1}k_{i+1} = b_i k_i$ we find

$$\alpha_i = a_i b_i k_i k^{-1} b_1^{-1}. \tag{7}$$

We have proved the claim. We can now easily show that $|D^i_{i+1} \cap \Gamma_{i+1}(y)| = b_i(b_1 - a_i - c_i)k_i k^{-1} b_1^{-1}$. Pick $v \in D^i_{i+1}$. It follows from the triangle inequality that $\partial(y,v) \in \{i-1, i, i+1\}$, so

$$|D^i_{i+1} \cap \Gamma_{i+1}(y)| = |D^i_{i+1}| - |D^i_{i+1} \cap \Gamma_i(y)| - |D^i_{i+1} \cap \Gamma_{i-1}(y)|.$$

Using Lemma 4.2(i), Lemma 4.3(i), Lemma 8.3 and (7) we find $|D^i_{i+1} \cap \Gamma_{i+1}(y)| = b_i(b_1 - a_i - c_i)k_i k^{-1} b_1^{-1}$. The result follows. ∎

Corollary 8.6 *With reference to Notation 3.1, for* $1 \le i \le D-1$ *we have*

$$E^*_1 A_{i+1} E^*_i A_{i+1} E^*_1 = b_i k_i k^{-1} E^*_1 + b_i k_i k^{-1} E^*_1 A E^*_1 + b_i(b_1 - a_i - c_i) k_i k^{-1} b_1^{-1} E^*_1 A_2 E^*_1.$$

Proof. For $y, z \in X$ we show that the (y,z)-entry of both sides are equal. If $y \notin \Gamma(x)$ or $z \notin \Gamma(x)$ then the (y,z)-entry of each side is 0. If $y, z \in \Gamma(x)$ then the (y,z)-entry of both sides are equal by Corollary 6.3, Lemma 6.5 and Lemma 8.5. The result follows. ∎

9 Some scalar products

With reference to Notation 3.1, in this section we compute some scalar products which we will need later.

Lemma 9.1 *With reference to Notation 3.1 let* W *denote an irreducible* T*-module with endpoint* 1*. Then* $JW = 0$*.*

Proof. Since W is not the primary module we have $E_0 W = 0$. Recall $J = |X| E_0$ so $JW = 0$. ∎

Lemma 9.2 *With reference to Notation 3.1 let* W *denote an irreducible* T*-module with endpoint* 1*. Then the following* (i), (ii) *hold for* $w \in E^*_1 W$*.*

(i) $E^*_i A_{i-1} w + E^*_i A_i w + E^*_i A_{i+1} w = 0$ *for* $1 \le i \le D-1$*.*

(ii) $E^*_D A_{D-1} w + E^*_D A_D w = 0$*.*

Proof. For each equation in Corollary 6.4 apply both sides to w and simplify using $E^*_1 w = w$ and Lemma 9.1. ∎

Corollary 9.3 *With reference to Notation 3.1 let* W *denote an irreducible* T*-module with endpoint* 1 *and local eigenvalue* η*. Then for* $w \in E^*_1 W$ *we have* $E^*_1 A_2 w = -(1+\eta) w$*.*

Proof. Set $i = 1$ in Lemma 9.2(i) and note that $E^*_1 A_0 w = w$ and $E^*_1 A w = \eta w$. ∎

Lemma 9.4 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue η. Then the following* (i)–(iii) *hold for $w \in E_1^*W$.*

(i) $\|E_i^*A_{i-1}w\|^2 = (b_1 - (c_i - 1)(1+\eta))c_ik_ik^{-1}b_1^{-1}\|w\|^2 \ \ (1 \leq i \leq D)$.

(ii) $\|E_i^*A_{i+1}w\|^2 = (k - b_i)(1+\eta)b_ik_ik^{-1}b_1^{-1}\|w\|^2 \ \ (1 \leq i \leq D-1)$.

(iii) $\langle E_i^*A_{i-1}w, E_i^*A_{i+1}w\rangle = -(1+\eta)c_ib_ik_ik^{-1}b_1^{-1}\|w\|^2 \ \ (1 \leq i \leq D-1)$.

Proof. (i) Evaluating $\|E_i^*A_{i-1}w\|^2 = \langle E_i^*A_{i-1}w, E_i^*A_{i-1}w\rangle$ using $E_1^*w = w$, line (1) and Corollary 8.2 we find

$$\|E_i^*A_{i-1}w\|^2 = \frac{c_ik_i}{k}\|w\|^2 + \frac{c_i(c_i-1)k_i}{kb_1}\langle w, E_1^*A_2w\rangle.$$

The result follows from this and Corollary 9.3.
(ii),(iii) Similar to the proof of (i) above. ∎

We now split the analysis into two cases, depending on whether W has local eigenvalue -1 or a_1.

10 The irreducible T-modules with endpoint 1 and local eigenvalue -1

With reference to Notation 3.1, in this section we describe the irreducible T-modules with endpoint 1 and local eigenvalue -1.

Lemma 10.1 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue -1. Then for $w \in E_1^*W$ and $1 \leq i \leq D-1$ we have $E_i^*A_{i+1}w = 0$.*

Proof. Immediate from Lemma 9.4(ii). ∎

Theorem 10.2 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue -1. Fix a nonzero $w \in E_1^*W$. Then the following is a basis for W:*

$$E_i^*A_{i-1}w \qquad\qquad (1 \leq i \leq D). \tag{8}$$

Proof. We first show that W is spanned by the vectors (8). Let W' denote the subspace of V spanned by the vectors (8) and note that $W' \subseteq W$. We claim that W' is a T-module. By construction W' is M^*-invariant. It follows from Lemma 7.2(i),(ii), Lemma 7.3(i),(ii), Lemma 7.4(i),(ii), Lemma 9.2, Lemma 10.1, $E_1^*w = w$ and $E_1^*Aw = -w$ that W' is invariant under each of L, F, R. Recall that $L + F + R = A$ and A generates M so W' is M-invariant. The claim follows. Note that $W' \neq 0$ since $w \in W'$ so $W' = W$ by the irreducibility of W. We now show that the vectors (8) are linearly independent. By (6) it suffices to show that $E_i^*A_{i-1}w \neq 0$ for $1 \leq i \leq D$. This follows from Lemma 9.4(i) and since $w \neq 0$. ∎

Corollary 10.3 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue* -1. *Then $E^*_i W$ has dimension* 1 *for* $1 \leq i \leq D$.

Proof. Immediate from Theorem 10.2. ∎

Corollary 10.4 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue* -1. *Then the following* (i), (ii) *hold.*

(i) *The dimension of W is D.*

(ii) *The diameter of W is $D-1$.*

Proof. Immediate from Corollary 10.3. ∎

Corollary 10.5 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue* -1. *Then $W = M^*Mw$ for all nonzero $w \in E^*_1 W$.*

Proof. By construction $M^*Mw \subseteq W$ and equality holds in view of Theorem 10.2. ∎

11 The irreducible T-modules with endpoint 1 and local eigenvalue -1: the A-action

With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1 and local eigenvalue -1. In this section we display the action of A on the basis for W given in Theorem 10.2. Since $A = L + F + R$ it suffices to give the actions of L, F, R on this basis.

Lemma 11.1 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue* -1. *Then the following* (i), (ii) *hold for all nonzero $w \in E^*_1 W$.*

(i) $Lw = 0$.

(ii) *For* $2 \leq i \leq D$,
$$LE^*_i A_{i-1} w = b_{i-1} E^*_{i-1} A_{i-2} w.$$

Proof. For each equation of Lemma 7.2(i),(ii) apply each side to w and simplify using $E^*_1 w = w$ and Lemma 10.1. ∎

Lemma 11.2 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue* -1. *Then the following holds for all nonzero $w \in E^*_1 W$ and* $1 \leq i \leq D$:
$$FE^*_i A_{i-1} w = (a_{i-1} + c_{i-1} - c_i) E^*_i A_{i-1} w.$$

Proof. For each equation of Lemma 7.3(i),(ii) apply each side to w and simplify using $E^*_1 w = w$, $E^*_1 Aw = -w$, Lemma 9.2 and Lemma 10.1. ∎

Lemma 11.3 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue* -1. *Then the following* (i), (ii) *hold for all nonzero* $w \in E_1^*W$.

(i) *For* $1 \leq i \leq D-1$,

$$RE_i^*A_{i-1}w = c_iE_{i+1}^*A_iw.$$

(ii) $RE_D^*A_{D-1}w = 0$.

Proof. For each equation of Lemma 7.4(i),(ii) apply each side to w and simplify using $E_1^*w = w$. ∎

12 The irreducible T-modules with endpoint 1 and local eigenvalue a_1

With reference to Notation 3.1, in this section we describe the irreducible T-modules with endpoint 1 and local eigenvalue a_1.

Lemma 12.1 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue* a_1. *For* $w \in E_1^*W$ *and* $2 \leq i \leq D-1$ *the determinant of*

$$\begin{pmatrix} \|E_i^*A_{i-1}w\|^2 & \langle E_i^*A_{i-1}w, E_i^*A_{i+1}w\rangle \\ \langle E_i^*A_{i+1}w, E_i^*A_{i-1}w\rangle & \|E_i^*A_{i+1}w\|^2 \end{pmatrix}$$

is equal to

$$c_ib_i(a_1+1)(a_i-a_1c_i)k_i^2k^{-1}b_1^{-2}\|w\|^4.$$

Proof. Evaluate the matrix entries using Lemma 9.4, take the determinant and simplify the result using (2). ∎

Theorem 12.2 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue* a_1. *Fix a nonzero* $w \in E_1^*W$. *Then the following is a basis for W:*

$$E_i^*A_{i-1}w \quad (1 \leq i \leq D), \qquad\qquad E_i^*A_{i+1}w \quad (2 \leq i \leq D-1). \tag{9}$$

Proof. We first show that W is spanned by the vectors (9). Let W' denote the subspace of V spanned by the vectors (9) and note that $W' \subseteq W$. We claim that W' is a T-module. By construction W' is M^*-invariant. It follows from Lemmas 7.2, 7.3, 7.4, Lemma 9.2, Corollary 9.3, $E_1^*w = w$ and $E_1^*Aw = a_1w$ that W' is invariant under each of L, F, R. Recall that $L+F+R = A$ and A generates M so W' is M-invariant. The claim follows. Note that $W' \neq 0$ since $w \in W'$ so $W' = W$ by the irreducibility of W.

We now show that the vectors (9) are linearly independent. By (6) and since $w \neq 0$, it suffices to show that $E_i^*A_{i-1}w$, $E_i^*A_{i+1}$ are linearly independent for $2 \leq i \leq D-1$, and that $E_D^*A_{D-1}w \neq 0$. For $2 \leq i \leq D-1$ the vectors $E_i^*A_{i-1}w$, $E_i^*A_{i+1}w$ are linearly independent since their matrix of inner products has nonzero determinant by Corollary

3.2 and Lemma 12.1. It follows from (2) and Lemma 9.4(i) that $\|E^*_D A_{D-1} w\|^2 = (a_D - a_1 c_D) c_D k_D k^{-1} b_1^{-1} \|w\|^2$. Now $E^*_D A_{D-1} w \neq 0$ by Corollary 3.2. By these comments the vectors (9) are linearly independent and the result follows. ∎

We emphasize an idea from the proof of Theorem 12.2.

Corollary 12.3 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue a_1. Fix a nonzero $w \in E^*_1 W$. Then the following* (i)–(iii) *hold.*

(i) *$E^*_1 W$ has a basis*

$$w.$$

(ii) *For $2 \leq i \leq D-1$ the subspace $E^*_i W$ has a basis*

$$E^*_i A_{i-1} w, \qquad\qquad E^*_i A_{i+1} w.$$

(iii) *$E^*_D W$ has a basis*

$$E^*_D A_{D-1} w.$$

Corollary 12.4 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue a_1. Then the following* (i)–(iii) *hold.*

(i) *$E^*_1 W$ has dimension* 1.

(ii) *$E^*_i W$ has dimension* 2 *for $2 \leq i \leq D-1$.*

(iii) *$E^*_D W$ has dimension* 1.

Proof. Immediate from Corollary 12.3. ∎

Corollary 12.5 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue a_1. Then the following* (i), (ii) *hold.*

(i) *The dimension of W is $2D-2$.*

(ii) *The diameter of W is $D-1$.*

Proof. Immediate from Corollary 12.4. ∎

Corollary 12.6 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint* 1 *and local eigenvalue a_1. Then $W = M^* M w$ for all nonzero $w \in E^*_1 W$.*

Proof. By construction $M^* M w \subseteq W$ and equality holds in view of Theorem 12.2. ∎

13 The irreducible T-modules with endpoint 1 and local eigenvalue a_1: the A-action

With reference to Notation 3.1 let W denote irreducible T-module with endpoint 1 and local eigenvalue a_1. In this section we display the action of A on the basis for W given in Theorem 12.2. Since $A = L + F + R$ it suffices to give the actions of L, F, R on this basis.

Lemma 13.1 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1 and local eigenvalue a_1. Then the following (i)–(v) hold for all nonzero $w \in E_1^*W$.*

(i) $Lw = 0$.

(ii) $LE_2^*Aw = (k - c_2(a_1+1))w$.

(iii) *For $3 \le i \le D$,*

$$LE_i^*A_{i-1}w = b_{i-1}E_{i-1}^*A_{i-2}w + (c_i - c_{i-1})E_{i-1}^*A_iw.$$

(iv) $LE_2^*A_3w = -b_2(a_1+1)w$.

(v) *For $3 \le i \le D-1$,*

$$LE_i^*A_{i+1}w = b_iE_{i-1}^*A_iw.$$

Proof. For each equation of Lemma 7.2, apply each side to w and simplify using $E_1^*w = w$ and Corollary 9.3. ∎

Lemma 13.2 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1 and local eigenvalue a_1. Then the following (i)–(iv) hold for all nonzero $w \in E_1^*W$.*

(i) $Fw = a_1w$.

(ii) *For $2 \le i \le D-1$,*

$$\begin{aligned} FE_i^*A_{i-1}w &= \big(a_{i-1} + a_1(c_i - c_{i-1}) - c_i(a_1+1)(b^i - b^{i-2})(b^i-1)^{-1}\big)E_i^*A_{i-1}w \\ &\quad - c_i(b^i - b^{i-2})(b^i-1)^{-1}E_i^*A_{i+1}w. \end{aligned}$$

(iii) $FE_D^*A_{D-1}w = \big(a_{D-1} + a_1(c_D - c_{D-1}) - c_D(a_1+1)(b^D - b^{D-2})(b^D-1)^{-1}\big)E_D^*A_{D-1}w$.

(iv) *For $2 \le i \le D-1$,*

$$FE_i^*A_{i+1}w = a_iE_i^*A_{i+1}w.$$

Proof. For each equation of Lemma 7.3, apply each side to w and simplify using $E_1^*w = w$, $E_1^*Aw = a_1w$ and Lemma 9.2. ∎

Lemma 13.3 *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1 and local eigenvalue a_1. Then the following* (i)–(iv) *hold for all nonzero $w \in E_1^* W$.*

(i) *For $1 \leq i \leq D-1$,*
$$RE_i^* A_{i-1} w = c_i E_{i+1}^* A_i w.$$

(ii) $RE_D^* A_{D-1} w = 0$.

(iii) *For $2 \leq i \leq D-2$,*
$$\begin{aligned} RE_i^* A_{i+1} w &= (a_1+1)(c_{i+1}(b^{i-1}-1)(b^{i+1}-1)^{-1} - c_i) E_{i+1}^* A_i w \\ &\quad + c_{i+1}(b^{i-1}-1)(b^{i+1}-1)^{-1} E_{i+1}^* A_{i+2} w. \end{aligned}$$

(iv) $RE_{D-1}^* A_D w = (a_1+1)(c_D(b^{D-2}-1)(b^D-1)^{-1} - c_{D-1}) E_D^* A_{D-1} w$.

Proof. For each equation of Lemma 7.4, apply each side to w and simplify using $E_1^* w = w$, $E_1^* A w = a_1 w$ and Lemma 9.2. ∎

14 The isomorphism class of an irreducible T-module with endpoint 1

With reference to Notation 3.1, in this section we prove that up to isomorphism there exist exactly two irreducible T-modules with endpoint 1.

Proposition 14.1 *With reference to Notation 3.1, any two irreducible T-modules with endpoint 1 and local eigenvalue -1 are isomorphic.*

Proof. Let W and W' denote irreducible T-modules with endpoint 1 and local eigenvalue -1. Fix nonzero $w \in E_1^* W$, $w' \in E_1^* W'$. By Theorem 10.2, W and W' have bases $\{E_i^* A_{i-1} w \mid 1 \leq i \leq D\}$ and $\{E_i^* A_{i-1} w' \mid 1 \leq i \leq D\}$, respectively. Let $\sigma : W \to W'$ denote the vector space isomorphism defined by $\sigma(E_i^* A_{i-1} w) = E_i^* A_{i-1} w'$ for $1 \leq i \leq D$. We show that σ is a T-module isomorphism. Since A generates M and $E_0^*, E_1^*, \ldots, E_D^*$ is a basis for M^*, it suffices to show that σ commutes with each of $A, E_0^*, E_1^*, \ldots, E_D^*$.
Using the assertion (iv) below the line (5) and the definition of σ we immediately find that σ commutes with each of $E_0^*, E_1^*, \ldots, E_D^*$. It follows from Lemmas 11.1–11.3 that σ commutes with each of L, F, R. Recall $A = L + F + R$ so σ commutes with A. The result follows. ∎

Proposition 14.2 *With reference to Notation 3.1, any two irreducible T-modules with endpoint 1 and local eigenvalue a_1 are isomorphic.*

Proof. Similar to the proof of Proposition 14.1. ∎

Corollary 14.3 *With reference to Notation 3.1 fix a nonzero $w \in E_1^* V$ which is orthogonal to $\sum_{y \in \Gamma(x)} \hat{y}$. Assume that w is an eigenvector for $E_1^* A E_1^*$. Then $M^* M w$ is an irreducible T-module with endpoint 1.*

Proof. Let H denote the subspace of V spanned by the irreducible T-modules with endpoint 1. By construction and Lemma 9.1 E_1^*H is the orthogonal complement of $\sum_{y\in\Gamma(x)}\hat{y}$ in E_1^*V. Hence $w \in E_1^*H$. Note that $Tw \subseteq H$ so Tw is the orthogonal direct sum of some irreducible T-modules of endpoint 1. Call these T-modules $W_1, W_2, \ldots, W_s$. We show $s = 1$. By construction and since $w \in E_1^*V$ there exist $w_i \in E_1^*W_i$ $(1 \le i \le s)$ such that

$$w = w_1 + w_2 + \cdots + w_s. \tag{10}$$

For $1 \le i \le s$ we have $w_i \neq 0$; otherwise $Tw \subseteq W_1 + \cdots + W_{i-1} + W_{i+1} + \cdots W_s$. We claim that the T-modules $W_1, W_2, \ldots, W_s$ are mutually isomorphic. To see this, recall that w is an eigenvector for $E_1^*AE_1^*$; let η denote the corresponding eigenvalue. Applying $E_1^*AE_1^*$ to each term in (10) and using $E_1^*AE_1^* \in T$ we find $E_1^*AE_1^*w_i = \eta w_i$ for $1 \le i \le s$. Therefore each of $W_1, W_2, \ldots, W_s$ has local eigenvalue η, so $W_1, W_2, \ldots, W_s$ are mutually isomorphic by Propositions 14.1, 14.2. We have proved the claim. We can now easily show that $s = 1$. Suppose $s \ge 2$. By construction there exists $t \in T$ such that $tw = w_1$. We have $w_1 = tw = tw_1 + \cdots + tw_s$ and $tw_i \in W_i$ for $1 \le i \le s$. Therefore $tw_i = 0$ for $2 \le i \le s$. Now $(t-I)E_1^*$ is zero on W_1 and nonzero on W_i for $2 \le i \le s$; this contradicts the fact that $W_1, W_2, \ldots, W_s$ are mutually isomorphic. We conclude $s = 1$. Now $Tw = W_1$ is an irreducible T-module with endpoint 1. The result follows since $Tw = M^*Mw$ by Corollary 10.5 and Corollary 12.6. ∎

With reference to Notation 3.1 recall that V is an orthogonal direct sum of irreducible T-modules. Let W denote an irreducible T-module. By the *multiplicity with which W appears in V* we mean the number of irreducible T-modules in this sum which are isomorphic to W. For example the primary module $M\hat{x}$ appears in V with multiplicity 1.

Theorem 14.4 *With reference to Notation 3.1, up to isomorphism there exist exactly two irreducible T-modules with endpoint 1. The first has local eigenvalue -1 and appears in V with multiplicity $ka_1(a_1+1)^{-1}$. The second has local eigenvalue a_1 and appears in V with multiplicity $b_1(a_1+1)^{-1}$.*

Proof. By Corollary 4.7 each irreducible T-module with endpoint 1 has local eigenvalue -1 or a_1. By Proposition 14.1 (resp. Proposition 14.2) any two irreducible T-modules with endpoint 1 and local eigenvalue -1 (resp. a_1) are isomorphic. For $\eta \in \{a_1, -1\}$ let μ_η denote the multiplicity with which an irreducible T-module with endpoint 1 and local eigenvalue η appears in V. We show that $\mu_\eta = ka_1(a_1+1)^{-1}$ if $\eta = -1$ and $\mu_\eta = b_1(a_1+1)^{-1}$ if $\eta = a_1$. Let H_η denote the subspace of V spanned by all the irreducible T-modules with endpoint 1 and local eigenvalue η. We claim that μ_η is equal to the dimension of $E_1^*H_\eta$. Observe that H_η is a T-module so it is an orthogonal direct sum of irreducible T-modules:

$$H_\eta = W_1 + W_2 + \cdots + W_m \qquad \text{(orthogonal direct sum)}, \tag{11}$$

where m is a nonnegative integer, and where $W_1, W_2, \ldots, W_m$ are irreducible T-modules with endpoint 1 and local eigenvalue η. Apparently m is equal to μ_η. We show m is equal to the dimension of $E_1^*H_\eta$. Applying E_1^* to (11) we find

$$E_1^*H_\eta = E_1^*W_1 + E_1^*W_2 + \cdots + E_1^*W_m \qquad \text{(direct sum)}. \tag{12}$$

Note that each summand on the right in (12) has dimension 1. It follows that m is equal to the dimension of $E_1^* H_\eta$ and the claim is proven. Recall that $\sum_{y \in \Gamma(x)} \hat{y}$ is an eigenvector for $E_1^* A E_1^*$ with eigenvalue a_1. Let U_η denote the set of those vectors in $E_1^* V$ that are eigenvectors for $E_1^* A E_1^*$ with eigenvalue η and that are orthogonal to $\sum_{y \in \Gamma(x)} \hat{y}$. By Corollary 4.6(ii) the dimension of U_η is $ka_1(a_1+1)^{-1}$ if $\eta = -1$ and $k(a_1+1)^{-1} - 1 = b_1(a_1+1)^{-1}$ if $\eta = a_1$. We now show $E_1^* H_\eta = U_\eta$. By (12) and Lemma 9.1 we find $E_1^* H_\eta \subseteq U_\eta$. Pick a nonzero $w \in U_\eta$. By Corollary 14.3 and definition of H_η we find $w \in E_1^* M^* M w \subseteq E_1^* H_\eta$ implying $U_\eta \subseteq E_1^* H_\eta$. It follows $U_\eta = E_1^* H_\eta$. The result now follows from these comments and the above claim. ∎

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